



Mathematical Harmonies

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What is music? When you hear a flutist, a signal is sent from her fingers to your ears. As the flute is played, it vibrates. The vibrations travel through the air and vibrate your eardrums. These vibrations are fast oscillations in air pressure, which your ear detects as sound.

The Basics

The simplest model of a musical sound is a sine wave, where the domain (x-axis) is time and the range (y-axis) is pressure.

$$P = A \sin(2\pi ft)$$

where:

- P pressure, in decibels or Pascals
- t time, in seconds
- A amplitude (height of the wave) or volume, in decibels or Pascals
- f frequency or pitch, in hertz.
- T period, in seconds is the duration of one wave. $T = 1/f$

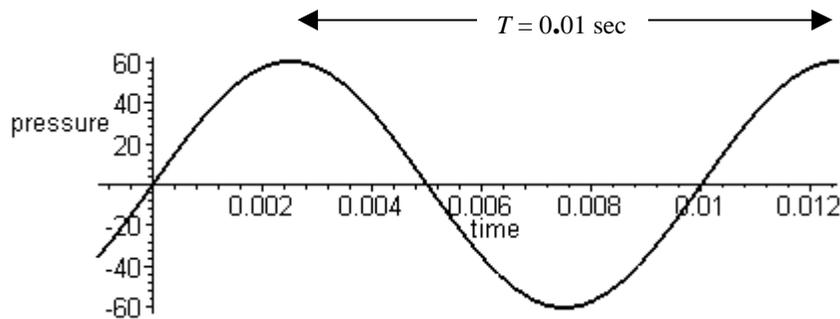


Figure 1. A sine wave with amplitude $A = 60$ dB and frequency $f = 100$ Hz.

In general, a sound has two characteristics: pitch and volume. The pitch, or note played, corresponds to the frequency of the wave. High notes have high frequencies, so the pressure varies quickly. Low notes have low frequencies. Frequency is measured in Hertz (Hz), which is the number of waves per second.

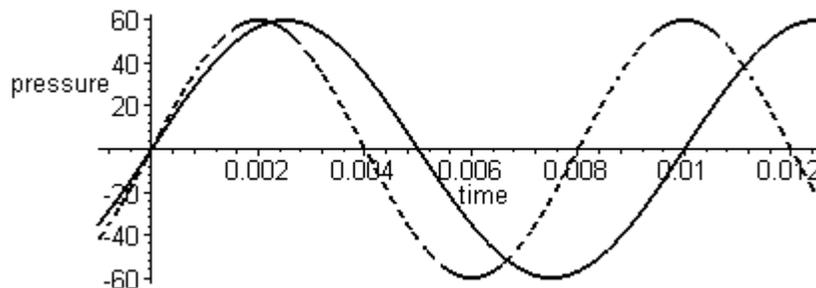


Figure 2. Two notes, both with amplitude $A = 60$ dB. The lower note has frequency $f = 100$ Hz (solid). The higher note has frequency $f = 125$ Hz (dashed).

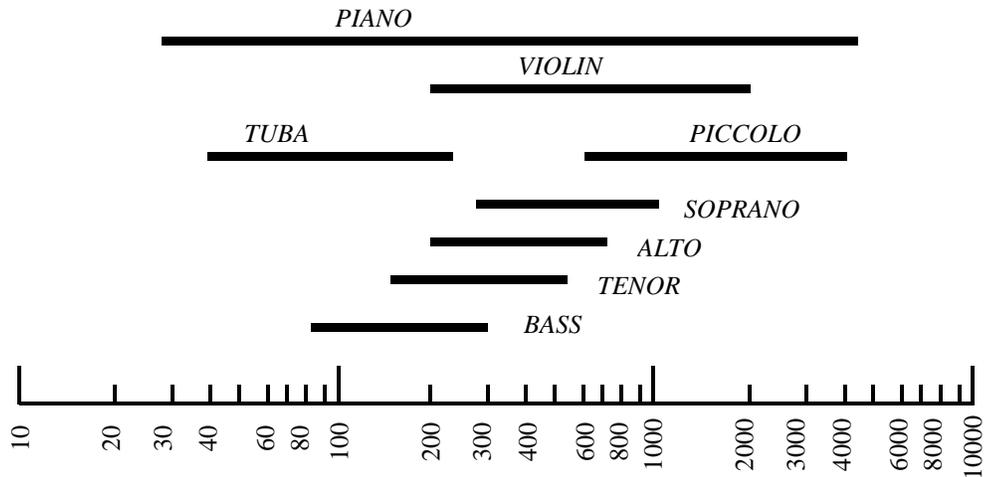


Figure 3. Frequency ranges of various instruments, in Hz. Audible frequencies range from 20 Hz to 20,000 Hz.

Volume, or loudness, corresponds to the amplitude of the pressure. When one hears loud music, like at a rock concert, the large pressure oscillations may be felt by the body.

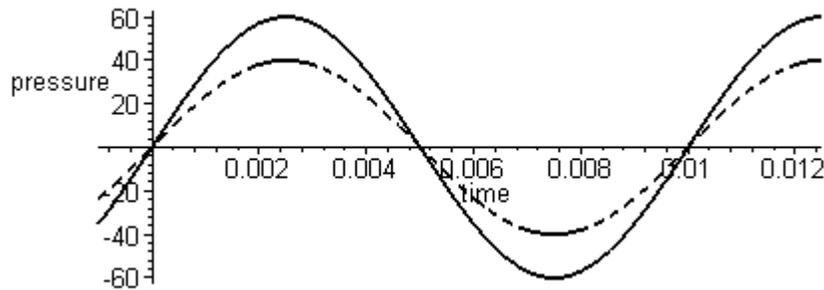


Figure 4. A loud note at $A = 60$ dB (solid) and a quiet note at $A = 40$ dB (dashed). Both notes have a frequency of $f = 100$ Hz.

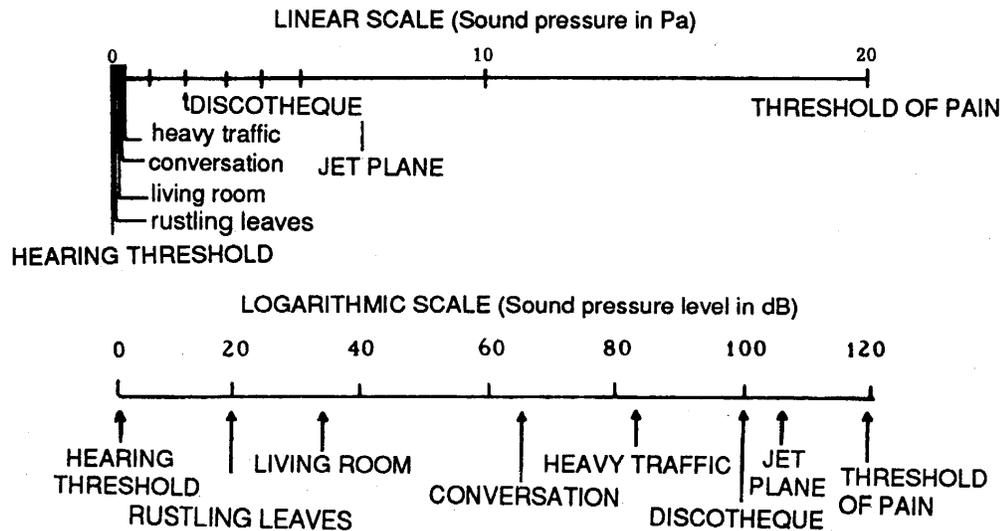


Figure 5. Intensities of various sounds on a linear and logarithmic scale.

Pressure is normally measured in Pascals, which is force per unit area ($1 \text{ Pa} = 1 \text{ N/m}^2$). As shown in Figure 5, most sounds are less than 1 Pa , while loud ones are between 5 and 10. The decibel scale is a log pressure scale, which is used for volume so that the quiet sounds are spread out. Pascals are converted to decibels as follows:

$$p_{dB} = 20 * \log \frac{P_{Pa}}{2 \times 10^{-5}}$$

The constant 2×10^{-5} was chosen because $2 \times 10^{-5} \text{ Pa}$ is considered the hearing threshold. This is where the p_{dB} is zero, because when $p_{Pa} = 2 \times 10^{-5} \text{ Pa}$, we have $p_{dB} = 20 * \log 1 = 0$.

Frequencies of Octaves and Harmonics

In order to understand why certain combinations of notes make harmony and others do not, we will study the simplest instrument, a single string. The formula for the frequency of a vibrating string is

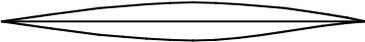
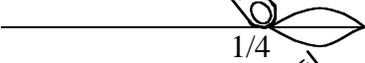
$$frequency = \frac{1}{2 * length} \sqrt{\frac{tension}{line\ density}}$$

where: frequency is in Hertz = 1/sec
length is in meters
tension is a force, in Newtons = $\text{kg} * \text{m}/\text{sec}^2$
line density is the string thickness, in kg/m

Notice that we may change the frequency, or pitch, in three ways:

- | | | | |
|--|----------------|-------------|-------------|
| 1. Tighten the string: | ↑ tension | results in: | ↑ frequency |
| 2. Use a thicker string: | ↑ line density | results in: | ↓ frequency |
| 3. Use fingers on frets ¹ : | ↓ length | results in: | ↑ frequency |

Specifically, frequency is inversely proportional to the length of the string. This means if I halve the length of the string, the frequency will double. It turns out that a doubled frequency is an octave higher. Using these facts, we may construct the following chart.

Note	Frequency	Diagram of vibrating string
low low low A	$f = 55 \text{ Hz}$	
low low A	$f = 110 \text{ Hz}$	
low A	$f = 220 \text{ Hz}$	
middle A	$f = 440 \text{ Hz}$	

¹ Frets are the vertical bars on the neck of a guitar.

Figure 6. Octaves of a vibrating string.

The sequence of frequencies of these octaves: 55, 110, 220, 440, is a geometric sequence. A geometric sequence is a sequence where the previous term is multiplied by a constant. In this case, the constant is two. A very simple example of a geometric sequence is 2, 4, 8, 16, 32. If this sequence were graphed, it would look like an exponential function.

The important point here is:

The frequencies of octaves form a geometric sequence.

This fact has many physical manifestations, such as:

- Low instruments must be much larger than high instruments. In general, an instrument which is an octave lower must be twice as large. For example, in the string family, as we progress from violin, viola, cello, to bass, the cello is large and the bass is very large.
- Organ pipes must also double in size to go down an octave. This is why the organ pipes at the front of a church, if arranged in descending order, approximate an exponential curve.
- Frets on a guitar are far apart at the neck and close together near the body, a pattern which also appears on log graphing paper. Frets and log paper both follow an inverse exponential pattern.

If we could watch our simple string vibrate with a slow motion camera, we would see that it vibrates in many modes, as shown below. The main mode is the fundamental frequency or first harmonic, and gives the note its specified frequency. The string may vibrate in higher modes, or harmonics, at various times or simultaneously.

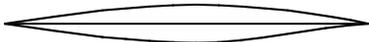
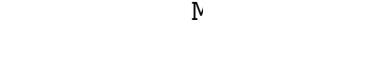
Note	Frequency	Harmonic	Diagram of vibrating string
low low low A	$f = 55 \text{ Hz}$	fundamental	
low low A	$f = 110 \text{ Hz}$	second	
low E	$f = 165 \text{ Hz}$	third	
low A	$f = 220 \text{ Hz}$	fourth	
middle C [#]	$f = 275 \text{ Hz}$	fifth	
middle E	$f = 330 \text{ Hz}$	sixth	
approx. middle G	$f = 385 \text{ Hz}$	seventh	
middle A	$f = 440 \text{ Hz}$	eighth	

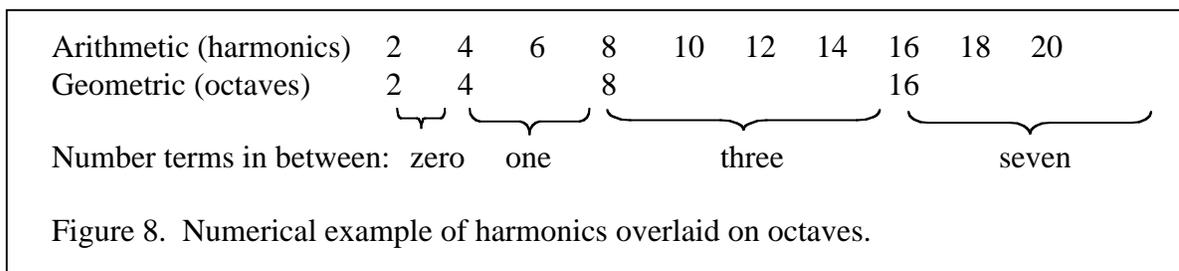
Figure 7. Harmonics of a vibrating string.

The sequence of frequencies of these harmonics: 55, 110, 165, 220, 275, \dots form an arithmetic sequence. An arithmetic sequence is a sequence where a constant is added to the previous term. In this case, the constant is 55. A simple example of an arithmetic sequence is 2, 4, 6, 8, 10, \dots

To summarize our important points,

*The frequencies of octaves form a geometric sequence.
The frequencies of harmonics form an arithmetic sequence.*

Let us overlay an arithmetic sequence (harmonics) on a geometric sequence (the octaves):



Notice that the number of arithmetic terms between each geometric is 0, 1, 3, 7, \dots Figure 9 shows the harmonics of low low low A, which have the same relation.

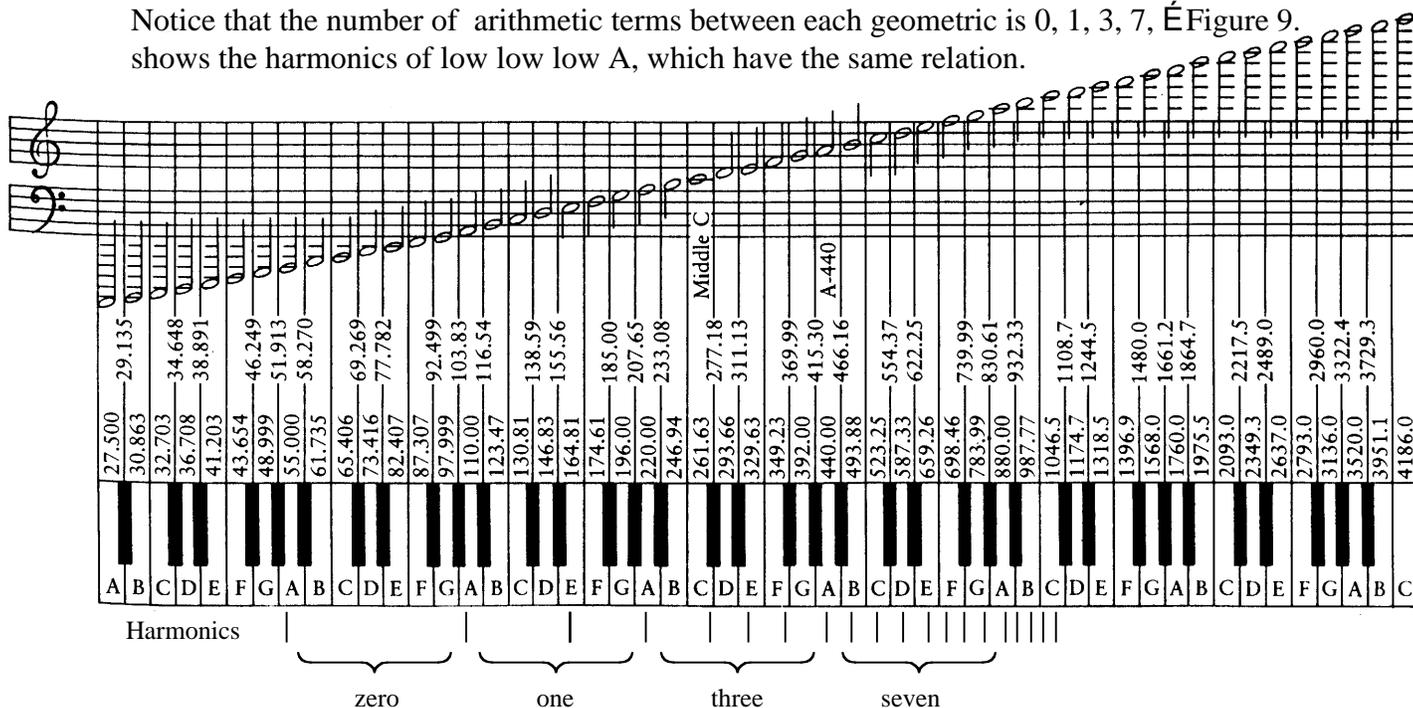


Figure 9. Harmonics of low low low A (as on Figure 7) shown as vertical lines below the keyboard. Frequencies are shown above the keyboard.

You may have noticed that the harmonics of A include C# and E, which are the notes of an A-major chord. We will return to this issue after some diversions.

Harmonics of Instruments

Two characteristics of a musical sound are volume and pitch. How does one know the difference between a flute and a violin, even when they play the same note and volume? If we measured the air pressure near a flute, oboe, and violin all playing middle A (440 Hz), it would look like this:

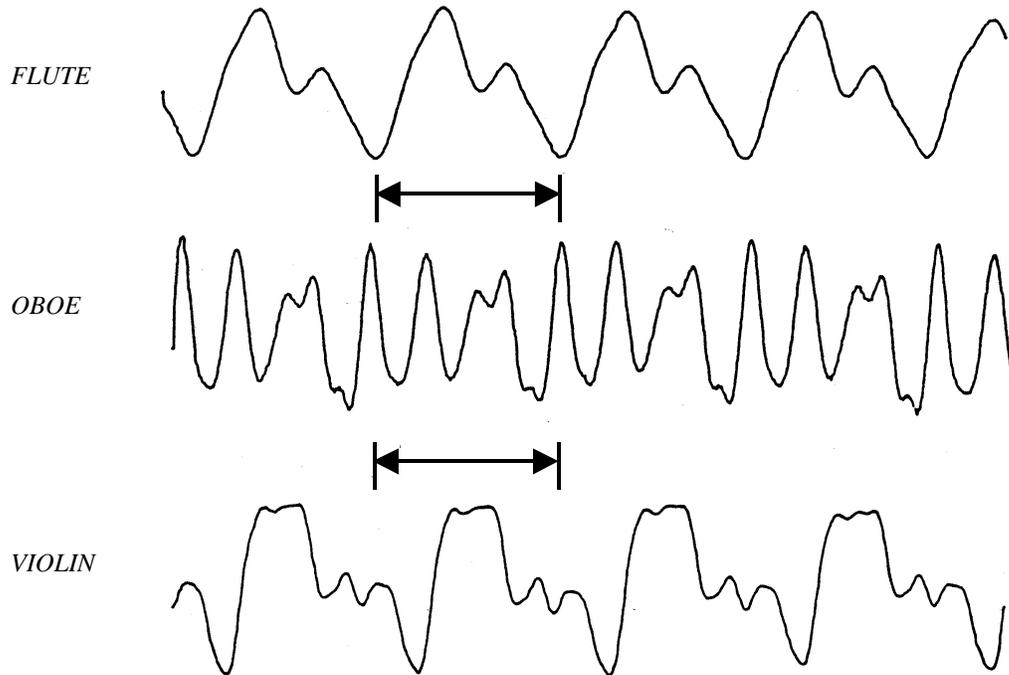


Figure 10. Pressure variations with time of a flute, oboe, and violin.

Their pressure signals look very different, even though the amplitude and fundamental frequencies are all the same. This difference is caused by the relative amplitudes of the higher harmonics. This can be seen when the volume of each harmonic is graphed separately, as follows.

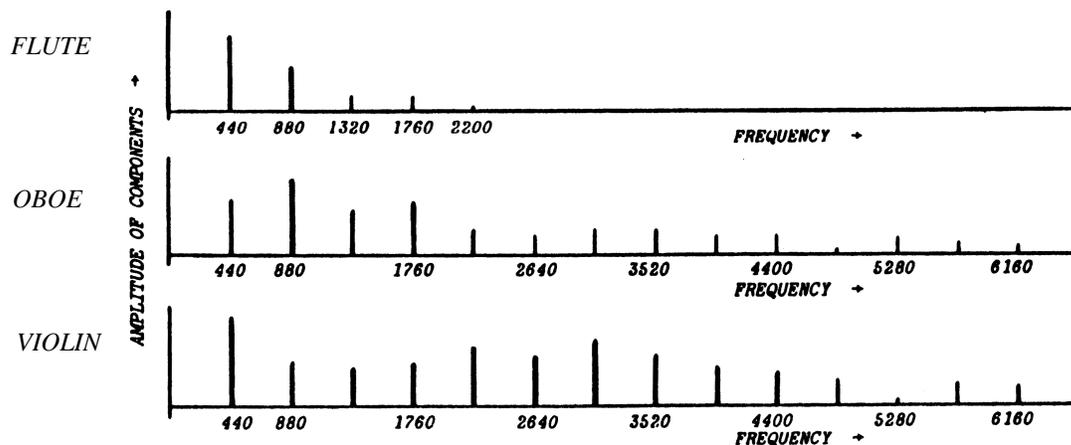


Figure 11. Amplitudes of the harmonics of a flute, oboe, and violin playing middle A².

² In advanced mathematics, these are called the Fourier coefficients of the wave forms in Figure 10. Fourier Analysis is used to calculate these coefficients for a given signal.

Notice that the flute's harmonics consist mostly of the fundamental at 440 Hz and the second harmonic at 880 Hz. When the air pressure near a flute is actually measured, we see the sum of these two harmonics. This is equivalent to adding the two sine curves as follows:

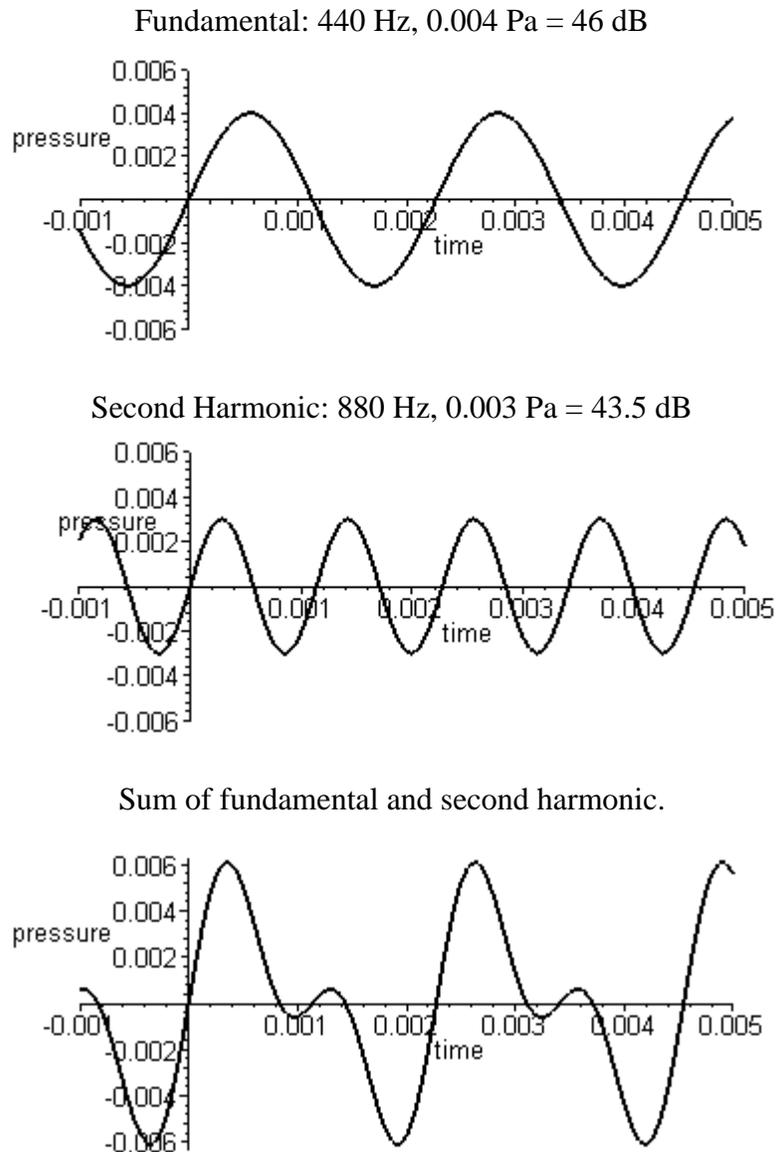


Figure 12. Summation of 1st and 2nd harmonic of a flute.

The third graph is the signature pressure wave of the flute (compare to Figure 10). The same process could be used to produce the oboe and violin pressure waves, but the other harmonics shown in Figure 11 must be added in.

Synthesized music imitates instruments by combining harmonics, just as we did for a flute above. Synthesized music often sounds fake because its harmonics are constant, while real music has harmonics that change subtly as the musician varies timbre, vibrato, and phrasing.

Beats and Intervals

When two sine waves are played with nearly the same frequency, beats are made. These beats can be heard by playing two guitar strings or flutes with one slightly flatter than the other.

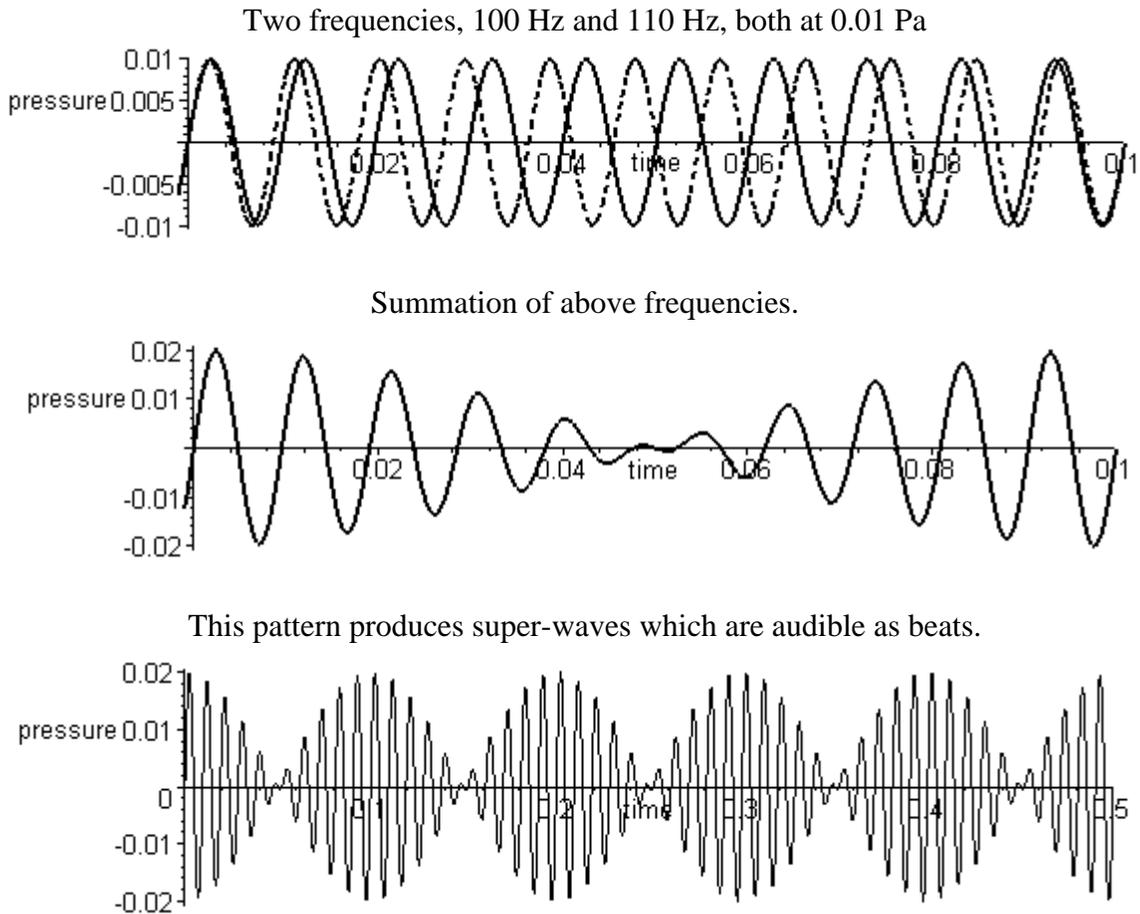


Figure 13. Superposition of two waves of slightly different frequency.

Notice how the two curves in the first graph vary between being aligned and in opposite alignment. The summation curve in the second graph is doubled when these two graphs are aligned and cancel out when they are in opposite alignment.

Beats are strongest when the frequency separation is between a half step and a minor third. When the separation is smaller than this, the beats are too slow for the ear to distinguish. When the separation is larger, the beats are too fast to hear. This is shown graphically in Figure 14.

For physiological reasons, the human mind dislikes beats. We may therefore assume that frequencies that are close enough to produce beats will not be harmonious. In fact, the strength of these beats can be used to represent the consonance, or harmoniousness, between two frequencies.

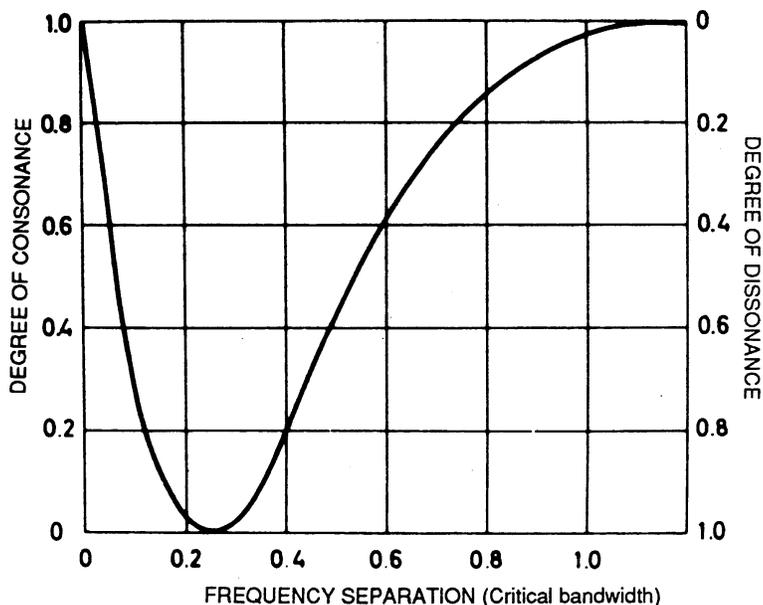


Figure 14. Consonance versus frequency separation (f_2/f_1)

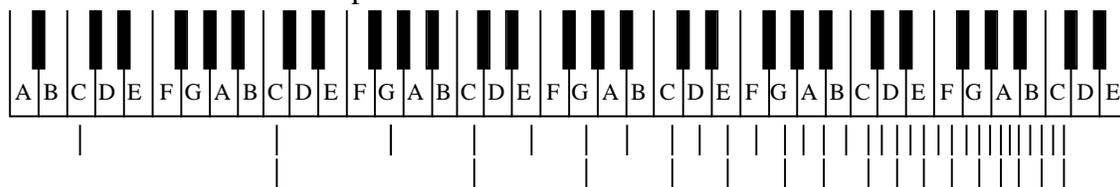
Our conclusion is

Frequencies close to each other create beats and sound bad (dissonance)

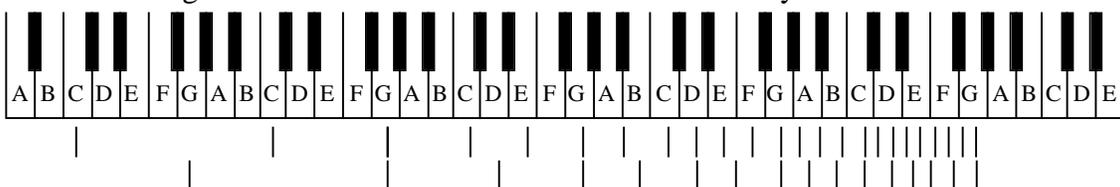
We may use this knowledge to investigate why certain combinations of notes sound harmonious and others do not. First we must cover some music vocabulary. An *interval* is the difference between two pitches. A *third* is an interval which is three steps above the bass. For example, in the key of C major, E is the third, and G is the fifth (see Figure 16).

Returning to our keyboard, let us examine the harmonics of several intervals.

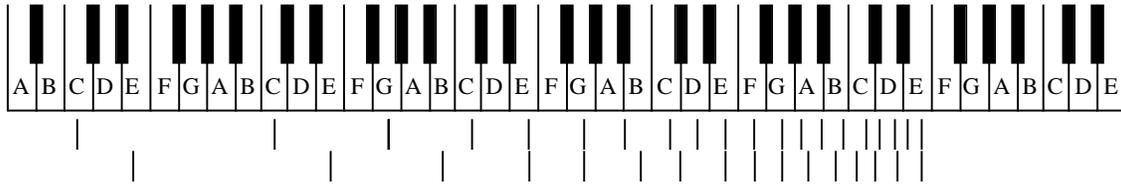
Octaves Harmonics of low C and low C. Octaves sound like the same note because all of their harmonics line up.



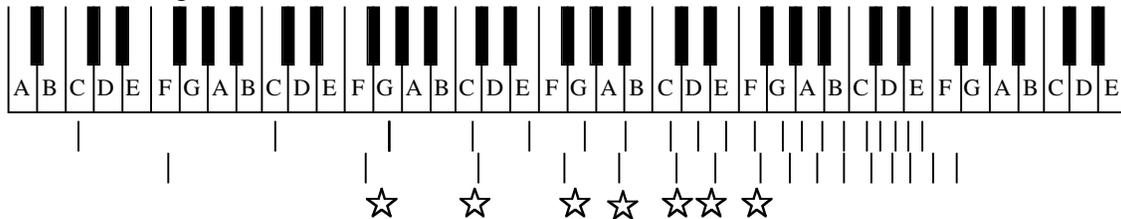
Fifth Harmonics of C and G. Here every other harmonic lines up while the others are not close enough to create beats. The interval of a fifth is very harmonious.



Third Harmonics of C and E. Many harmonics line up and most are not close enough to create beats. The interval of a third is also harmonious.



Diminished Fifth Harmonics of C and F[#]. Notice that no harmonics lines up and many are close enough to create beats. This interval is dissonant (*not* harmonious).



We can measure the dissonance of a pair of notes, like the C and F[#] above, as follows: Look for all the harmonics of C that are within a half step of a harmonic of F[#], but don't line up exactly. These are starred above. The dissonance is high for a pair of notes that have many of these close harmonics.

Because most of the harmonics of a third and fifth line up, these intervals have low dissonance. The diminished fifth has high dissonance. Figure 15 shows a graph which was constructed by testing many intervals for dissonance in this manner.

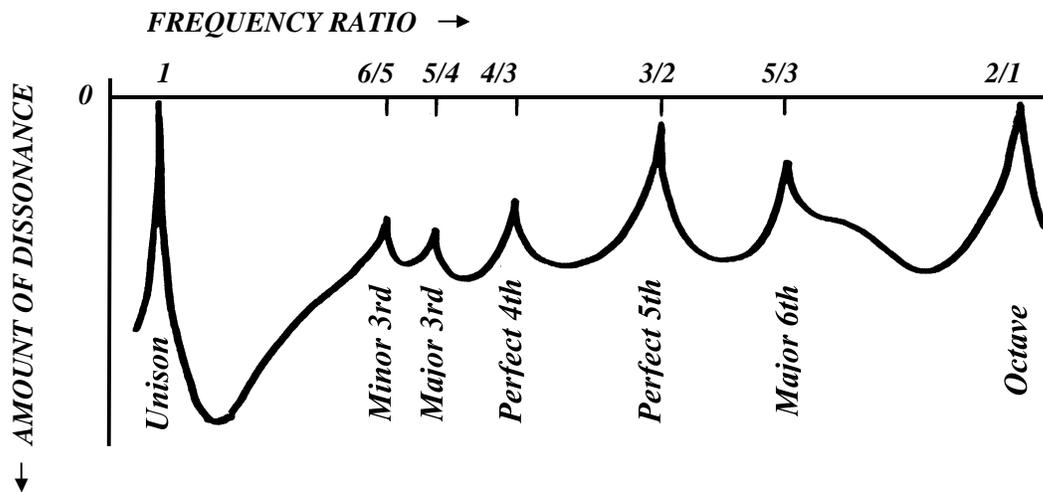


Figure 15. Total dissonance of intervals along an octave.

We have just shown that the major scale can be developed mathematically! Although cultures of the world have many different scales, they all include some combination of these intervals. Ancient cultures sang and played these intervals intuitively without knowing about frequencies and harmonics.

Just and Equal Temperament

Because of the way the harmonics line up the frequency ratios of major intervals turn out to be exact fractions, as shown in Figure 15. For example, the frequency of G is exactly $3/2$ times that of C. Beginning with a frequency of 65.4 Hz for C, we may build the major scale using these ratios.

Interval from C:		2 nd	3 rd	4 th	5 th	6 th	7 th	octave
	C	D	E	F	G	A	B	C
Frequency ratio:	1	9/8	5/4	4/3	3/2	5/3	15/8	2
Frequency (Hz):	65.4	73.6	81.8	87.2	98.1	109	122.6	130.8
Ratio from one note to the next:		1.125	(1.067)		1.111		(1.067)	
			1.111		1.125		1.125	

Figure 16. Frequencies of notes in key of C in just temperament. Parenthesis indicate half-steps.

This system of tuning is called *just temperament* because when these intervals are played the harmonics line up and it sounds *just perfect*.

But perfection comes at a price. Notice that the whole note frequency ratios are 1.125 for CD, FG and AB, but 1.111 for DE and GA. Suppose you had a flute tuned in just temperament in the key of C. Then the harmonics line up perfectly in the key of C, but not in any other key. For example, in the key of D the first ratio is 1.111, but it needs to be 1.125. With just temperament, your flute is only good in one key!

Just temperament instruments were the standard until the 1700s. A flutist would have had to own several flutes each tuned to a different key. Likewise, harpsichords had to have several keyboards for different keys. Vocalists and stringed instruments without frets are unaffected, because the pitch is not hardwired into the instrument but may be chosen exactly.

In the 18th century Bach and other musicians advocated a new tuning standard. In *equal temperament* the ratio for each step is always the same. The harmonics of an equal tempered instrument do not exactly line up. This is a small sacrifice, as only trained musicians can hear the difference. In return for *almost perfect* we get instruments that can play in every key. Modern instruments use equal temperament.

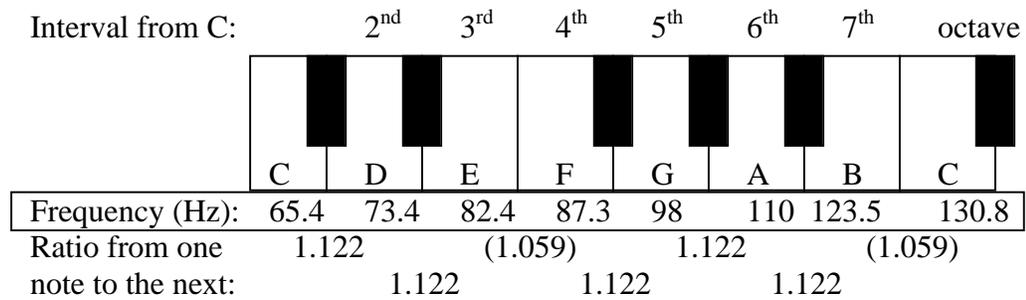


Figure 17. Frequencies of notes in key of C in equal temperament.

In equal temperament the frequency ratio for a whole step is always 1.122, which is between the 1.125 and 1.111 found in the just temperament scale. This number is arrived at as follows.

- There are 12 half steps in an octave, and an octave frequency ratio is 2.
The frequency ratio of each half step is: $\sqrt[12]{2} = 1.059$
- There are 6 whole steps in an octave.
The frequency ratio of each whole step is: $\sqrt[6]{2} = 1.122$

Conclusion

The mathematics of harmonics and vibrating strings is a beautiful example of the mathematics that surrounds us every day. Music, one of our most ancient and universal traditions, is at once quantifiable and emotional, both mathematical and moving. My hope is that this introduction can show students that math is not dry and boring, but exciting, intriguing, and fun.

References

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 Sundberg, Johan, *The Science of Musical Sounds*, Academic Press, San Diego, 1991

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amath.colorado.edu/outreach/demos/music

Figures: 3. Johnston p. 36 5. Sundberg p. 21 9. Pierce p. 21 10. Johnston p. 100
 11. Johnston p. 107 14. Sundberg p. 73 15. Johnston p. 256
 Graphs made with Maple 5.5, document in MS Word 97.