3.2 Vector and Tensor Mathematics

The variables used to describe physical quantities are of a number of types, including scalars, vectors, and tensors. Effort has been made in the notes to indicate these types consistently as follows:

$$s = \text{scalar, lightface italic}$$

$$\underline{\mathbf{v}} = \text{vector, boldface with single underscore}$$
$$\underline{\underline{\mathbf{T}}} = \text{tensor, boldface with double underscore} . \quad (3.10)$$

Scalars are used to represent physical quantities with no directional qualities, such as temperature, volume, and time. Vectors are used for quantities which have a single directional quality such as velocity and force. Tensors (we will consider only second-order tensors) are associated with quantities which have two directional characteristics, such as a momentum flux.

3.2.1 Vectors and Vector Operations

Given a coordinate system in three dimensions, a vector may thus be represented by an ordered set of three components which represent its projections v_1, v_2, v_3 on the coordinate axes $1, 2, 3$:

$$\underline{\mathbf{v}} = [v_1, v_2, v_3] . \quad (3.11)$$

The three most commonly used coordinate systems are rectangular, cylindrical, and spherical, as described in the Coordinate Systems Notebook. Alternatively, a vector may be represented by the sum of the magnitudes of its projections on three mutually perpendicular axes:

$$\underline{\mathbf{v}} = v_1\underline{\delta}_{\mathbf{1}} + v_2\underline{\delta}_{\mathbf{2}} + v_3\underline{\delta}_{\mathbf{z3}} = \sum_{i=1}^{3} v_i\underline{\delta}_{\mathbf{i}} . \quad (3.12)$$

The unit vectors $\underline{\delta}_{\mathbf{1}}, \underline{\delta}_{\mathbf{2}}, \underline{\delta}_{\mathbf{3}}$ are $\underline{\delta}_{\mathbf{x}}, \underline{\delta}_{\mathbf{y}}, \underline{\delta}_{\mathbf{z}}$ in the rectangular coordinate system, $\underline{\delta}_{\mathbf{r}}, \underline{\delta}_{\theta}, \underline{\delta}_{\mathbf{z}}$ in cylindrical coordinates and $\underline{\delta}_{\mathbf{r}}, \underline{\delta}_{\theta}, \underline{\delta}_{\phi}$ in spherical. Each of these unit vectors points in the direction of the indicated spatial coordinate, and has a magnitude of exactly one. The formulas for the vector and tensor operations described below are generally applicable to these three coordinate systems, *with the exception of differential operators*. Differential operators in cylindrical and spherical coordinates must be handled more explicitly because in those cases $\underline{\delta}_{\mathbf{i}}$ are not constant in direction (with the sole exception of $\underline{\delta}_{\mathbf{z}}$ in cylindrical coordinates).

The magnitude of a vector is given by:

$$|\underline{\mathbf{v}}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\sum_{i=1}^{3} v_i^2} \ . \tag{3.13}$$

Addition and subtraction of vectors is easily executed:

$$\underline{\mathbf{v}} + \underline{\mathbf{w}} = (v_1 + w_1)\underline{\delta}_{\mathbf{1}} + (v_2 + w_2)\underline{\delta}_{\mathbf{2}} + (v_3 + w_3)\underline{\delta}_{\mathbf{3}} = \sum_{i=1}^{3}(v_i + w_i)\underline{\delta}_{\mathbf{i}} \ , \tag{3.14}$$

as well as multiplication by a scalar

$$s\underline{\mathbf{v}} = (sv_1)\underline{\delta}_{\mathbf{1}} + (sv_2)\underline{\delta}_{\mathbf{2}} + (sv_3)\underline{\delta}_{\mathbf{3}} = s\sum_{i=1}^{3} v_i\underline{\delta}_{\mathbf{i}} \ . \tag{3.15}$$

The dot product of two vectors results in a scalar:

$$\underline{\mathbf{v}} \bullet \underline{\mathbf{w}} = v_1w_1 + v_2w_2 + v_3w_3 = \sum_{i=1}^{3} v_iw_i \ . \tag{3.16}$$

3.2.2 Tensors and Tensor Operations

A tensor is similarly represented by an ordered array of nine components:

$$\underline{\underline{\mathbf{T}}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & t_{23} & T_{33} \end{bmatrix} \ . \tag{3.17}$$

The diagonal elements of a tensor are those which have two identical subscripts, while the other elements are termed nondiagonal. The transpose of a tensor is obtained by interchanging the subscripts on each element:

$$\underline{\underline{\mathbf{T}}}^T = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{32} & T_{33} \end{bmatrix} \ . \tag{3.18}$$

A tensor is described as symmetric when $\underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}^T$. One special tensor is the unit tensor:

$$\underline{\underline{\delta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{3.19}$$

The dyadic product of two vectors results in a tensor, as follows:

$$\underline{\mathbf{v}}\ \underline{\mathbf{w}} = \begin{bmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{bmatrix} . \tag{3.20}$$

This leads to the definition of the unit dyads, of which there are nine:

$$\underline{\delta}_{\mathbf{1}}\ \underline{\delta}_{\mathbf{1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \tag{3.21}$$

$$\underline{\delta}_{\mathbf{1}}\ \underline{\delta}_{\mathbf{2}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , etc. \tag{3.22}$$

In a similar manner to vectors, tensors are easily added

$$\underline{\underline{\mathbf{T}}} + \underline{\underline{\mathbf{U}}} = \begin{bmatrix} T_{11}+U_{11} & T_{12}+U_{12} & T_{13}+U_{13} \\ T_{21}+U_{21} & T_{22}+U_{22} & T_{23}+U_{23} \\ T_{31}+U_{31} & T_{32}+U_{32} & T_{33}+U_{33} \end{bmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}(T_{ij}+U_{ij})\underline{\delta}_{\mathbf{i}}\underline{\delta}_{\mathbf{j}} , \tag{3.23}$$

or multiplied by scalars:

$$s\underline{\underline{\mathbf{T}}} = \begin{bmatrix} sT_{11} & sT_{12} & sT_{13} \\ sT_{21} & sT_{22} & sT_{23} \\ sT_{31} & sT_{23} & sT_{33} \end{bmatrix} = s\sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\underline{\delta}_{\mathbf{i}}\underline{\delta}_{\mathbf{j}} . \tag{3.24}$$

The double dot product of two tensors results in a scalar:

$$\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{U}}} = \begin{matrix} T_{11}U_{11}+ & T_{12}U_{21}+ & T_{13}U_{31}+ \\ T_{21}U_{12}+ & T_{22}U_{22}+ & T_{23}U_{32}+ \\ T_{31}U_{13}+ & T_{32}U_{23}+ & T_{33}U_{33} \end{matrix} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}U_{ji} . \tag{3.25}$$

The dot product of a tensor with a vector is:

$$\underline{\underline{\mathbf{T}}} \bullet \underline{\mathbf{v}} = \begin{matrix} \underline{\delta}_{\mathbf{1}}(T_{11}v_1 + T_{12}v_2 + T_{13}v_3) & + \\ \underline{\delta}_{\mathbf{2}}(T_{21}v_1 + T_{22}v_2 + T_{23}v_3) & + \\ \underline{\delta}_{\mathbf{3}}(T_{31}v_1 + T_{32}v_2 + T_{33}v_3) & \end{matrix} = \sum_{i=1}^{3}\underline{\delta}_{\mathbf{i}}\left(\sum_{j=1}^{3} T_{ij}v_j\right) . \tag{3.26}$$

In contrast, the dot product of a vector with a tensor is:

$$\underline{\mathbf{v}} \bullet \underline{\underline{\mathbf{T}}} = \begin{array}{ll} \underline{\delta}_{\mathbf{1}}(v_1 T_{11} + v_2 T_{21} + v_3 T_{31}) & + \\ \underline{\delta}_{\mathbf{2}}(v_1 T_{12} + v_2 T_{22} + v_3 T_{32}) & + \\ \underline{\delta}_{\mathbf{3}}(v_1 T_{13} + v_2 T_{23} + v_3 T_{33}) & \end{array} = \sum_{i=1}^{3} \underline{\delta}_{\mathbf{i}} \left(\sum_{j=1}^{3} v_j T_{ji} \right) . \quad (3.27)$$

In general, $(\underline{\underline{\mathbf{T}}} \bullet \underline{\mathbf{v}}) \neq (\underline{\mathbf{v}} \bullet \underline{\underline{\mathbf{T}}})$, however, they are equal if $\underline{\underline{\mathbf{T}}}$ is symmetric.

The magnitude of a tensor is defined as:

$$|\underline{\underline{\mathbf{T}}}| = \sqrt{\frac{1}{2}(\underline{\underline{\mathbf{T}}}:\underline{\underline{\mathbf{T}}}^T)} = \sqrt{\frac{1}{2} \sum_i \sum_j T_{ij}^2} . \quad (3.28)$$

3.2.3 Further Reading

A simple introduction to vectors and tensors is provided by:

- H. Anton, **Elementary Linear Algebra, 4th Ed.**, John Wiley and Sons, New York (1984).

Numerous problems, some with solutions may be found in:

- M.R. Spiegel **Vector Analysis**, Schaum's Outline Series, McGraw-Hill Book Company (1959).

- F. Ayres, **Matrices**, Schaum's Outline Series, McGraw-Hill Book Company (1962).

An excellent discussion of vector and tensor notation which is particularly relevant to polymer processing is in Appendix A of:

- R.B. Bird, R.C. Armstrong, O. Hassager, **Dynamics of Polymer Liquids, Vol. 1**, John Wiley and Sons, New York (1987).

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